

# A NOTE ON GENERIC OBJECTS AND LOCALLY FINITE TRIANGULATED CATEGORIES

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**ABSTRACT.** We show that the homotopy category of injective  $A$ -modules is generically trivial if and only if the derived category of all modules is generically trivial for an algebra  $A$ . Moreover we show some connections between the generic objects, locally finiteness and Krull-Gabriel dimension.

## 1. INTRODUCTION

Locally finiteness is an important finiteness condition on a triangulated category [1, 3, 15, 18]. There are some other approaches to view the locally finite triangulated category. We shall explore the connections between these finiteness conditions on triangulated categories.

Recall that an object  $C$  in a triangulated category  $\mathcal{T}$  with arbitrary coproduct is called *compact* if the functor  $\mathrm{Hom}_{\mathcal{T}}(C, -)$  commutes with coproducts. The triangulated category  $\mathcal{T}$  is *compactly generated* if there is exists a set  $T$  of compact objects of  $\mathcal{T}$  such that  $\mathrm{Hom}(T, X) = 0$  implies that  $X = 0$ . If a triangulated category is compactly generated, then its subcategory  $\mathcal{T}^c$  of all compact objects is also a triangulated category. For example, the unbounded derived category  $D(\mathrm{Mod} R)$  of all modules for neotherian ring  $R$  is compactly generated with  $D(\mathrm{Mod} R)^c$  the category of all perfect complexes.

Generic objects in derived categories were studied in [4, 10]. Now we consider generic objects in compactly generated triangulated categories. Generic objects are an indecomposable non-compact object satisfying some finiteness condition. The appearance of generic objects determines the properties of the triangulated category. A triangulated category is called *generically trivial* if it does not contain any generic object.

By [12],  $D(\mathrm{Mod} A)$  is generically trivial iff the algebra  $A$  is derived equivalent to an algebra of Dynkin type. Now we consider when the category  $K(\mathrm{Inj} A)$  is generically trivial. A compactly generated triangulated category  $\mathcal{T}$  is *associated with an algebra*  $A$  if it is of the form either  $K(\mathrm{Inj} A)$  or  $D(\mathrm{Mod} A)$  or  $\underline{\mathrm{Mod}} A$  with  $A$  a self-injective algebra. The *abelianisation*  $Ab(\mathcal{D})$  of a triangulated category  $\mathcal{D}$  is the category of all additive functors  $F : \mathcal{D}^{op} \rightarrow \mathrm{Ab}$  into the category of abelian groups satisfying

$$\mathrm{Hom}_{\mathcal{D}}(-, X) \rightarrow \mathrm{Hom}_{\mathcal{D}}(-, Y) \rightarrow F \rightarrow 0,$$

for  $X$  and  $Y$  in  $\mathcal{D}$ . We have the following main theorem. The result is closely related to compactly generated triangulated categories of finite type in [5].

*Theorem 1.1.* Let  $\mathcal{T}$  be a compactly generated triangulated category associated with an algebra, set  $\mathcal{A} = Ab(\mathcal{T}^c)$  then the following are equivalent.

- (1)  $\mathcal{T}$  is generically trivial.
- (2)  $\mathcal{T}^c$  is locally finite.
- (3)  $\mathrm{KG} \dim \mathcal{A} = 0$ .

(4)  $\mathcal{A}$  is locally noetherian.

In section 2, we give some introductions about endofinite objects and the recollement between  $K(\text{Inj } A)$  and  $D(\text{Mod } A)$ . In section 3, we give an example of generically trivial triangulated category. Let  $A$  be a self-injective algebra, the stable category  $\underline{\text{Mod}} A$  of all  $A$ -modules is a compactly generated triangulated category. The category  $\underline{\text{Mod}} A$  is generically trivial iff  $A$  is of finite type. In section 4, we consider the generic objects in the category  $K(\text{Inj } A)$  for an algebra  $A$ . We show that  $K(\text{Inj } A)$  is generically trivial iff the derived category  $D(\text{Mod } A)$  is generically trivial. In section 5, we consider the Krull-Gabriel dimension of  $\text{Ab}(\mathcal{D})$  and show the connections between generically trivial, locally finiteness and Krull-Gabriel dimension.

In the paper, we assume that  $k$  is an algebraically closed field and  $A$  is a finite dimensional  $k$ -algebra with a connected quiver  $Q$ .

## 2. PRELIMINARY

In this section, we introduce the endofinite object in a compactly generated category and the relation between  $K(\text{Inj } A)$  and  $D(\text{Mod } A)$ .

**2.1. Endofinite objects.** For a triangulated category  $\mathcal{T}$ , we mean  $\mathcal{T}$  is  $k$ -linear, i.e  $\text{Hom}_{\mathcal{T}}(X, Y)$  is a  $k$ -vector space. For a triangulated category  $\mathcal{T}$  with arbitrary coproducts, an object  $C$  of  $\mathcal{T}$  is *compact* if the functor  $\text{Hom}_{\mathcal{T}}(C, -)$  commutes with coproducts. The triangulated category  $\mathcal{T}$  is *compactly generated* if there exists a set  $T$  of compact objects of  $\mathcal{T}$  such that  $\text{Hom}(T, X) = 0$  implies that  $X = 0$ . The full subcategory  $\mathcal{T}^c$  of all compact objects of  $\mathcal{T}$  is a triangulated category.

Let  $\text{Mod } A$  be the category of all  $A$ -modules and  $\text{Inj } A$  is the full additive subcategory of all injective  $A$ -modules. The unbounded derived category  $D(\text{Mod } A)$  and the homotopy category  $K(\text{Inj } A)$  both are compactly generated. The category  $D(\text{Mod } A)^c$  is equivalent to the homotopy category  $K^b(\text{proj } A)$  of finitely generated projective  $A$ -modules. The category  $K(\text{Inj } A)^c$  is equivalent to the bounded derived category  $D^b(\text{mod } A)$  of the module category  $\text{mod } A$ , the category of finitely generated  $A$ -modules.

*Definition 2.1.* Let  $\mathcal{T}$  be a compactly generated triangulated category. An object  $E$  in  $\mathcal{T}$  is endofinite if the  $\text{End}_{\mathcal{T}} E$ -module  $\text{Hom}(X, E)$  has finite length for any  $X$  in  $\mathcal{T}^c$ .

For a compactly generated triangulated category  $\mathcal{T}$ , a full triangulated subcategory  $\mathcal{S}$  of  $\mathcal{T}$  is *localizing* if  $\mathcal{S}$  is closed under taking coproducts. The category  $D(\text{Mod } A)$  could be viewed as the localizing subcategory of  $K(\text{Inj } A)$  which is generated by the injective resolution of  $A$ . The relation between endofinite objects in a triangulated category  $\mathcal{T}$  and its localizing subcategory is established by the following result.

*Lemma 2.2.* [13, Lemma 1.3] Let  $\mathcal{S}$  be a localizing subcategory of  $\mathcal{T}$  which is generated by compact objects from  $\mathcal{T}$ , and  $q : \mathcal{T} \rightarrow \mathcal{S}$  be a right adjoint of the inclusion  $i : \mathcal{S} \rightarrow \mathcal{T}$ .

- (1)  $\mathcal{S}$  is a compactly generated triangulated category.
- (2) If  $X$  is an endofinite object in  $\mathcal{T}$ , then  $q(X)$  is endofinite in  $\mathcal{S}$ .

By this lemma, we know that for each endofinite object in  $K(\text{Inj } A)$  there is a corresponding endofinite object in  $D(\text{Mod } A)$ .

**2.2. Recollement.** The close relation between  $D(\text{Mod } A)$  and  $K(\text{Inj } A)$  is contained in the following recollement. We recall the recollement and summarize how to construct it [14].

$$K_{ac}(\text{Inj } A) \begin{array}{c} \xleftarrow{I_\rho} \\ \xrightarrow{I_\lambda} \end{array} K(\text{Inj } A) \begin{array}{c} \xleftarrow{L_\rho} \\ \xrightarrow{L_\lambda} \end{array} D(\text{Mod } A) .$$

Consider the canonical functors  $I : K_{ac}(\text{Inj } A) \rightarrow K(\text{Inj } A)$  and  $L : K(\text{Inj } A) \xrightarrow{inc} K(\text{Mod } A) \xrightarrow{can} D(\text{Mod } A)$ , we should show that  $I$  and  $L$  both have right and left adjoints.

Firstly, we show that  $L$  has right adjoint  $L_\rho$ . This is equivalent to the functor  $I$  has right adjoint  $I_\rho$  [14, Lemma 3.2]. Let  $K_{inj}(A)$  be the smallest triangulated category of  $K(\text{Mod } A)$  closed under taking products and contains  $\text{Inj } A$ . The inclusion functor  $K_{inj}(A) \rightarrow K(\text{Inj } A)$  preserves products, and has a left adjoint  $i : K(\text{Mod } A) \rightarrow K_{inj}(A)$  by [16, Theorem 8.6.1]. The functor  $i$  induces an equivalence

$$D(\text{Mod } A) \xrightarrow{\sim} K_{inj}(A).$$

By the natural isomorphism  $\text{Hom}_{D(\text{Mod } A)}(X, Y) \cong \text{Hom}_{K(\text{Mod } A)}(X, iY)$ , we can take the right adjoint  $L_\rho$  of  $L$  as the composition

$$D(\text{Mod } A) \xrightarrow{i} K_{inj}(A) \longrightarrow K(\text{Inj } A) .$$

Let  $\mathcal{K}$  be the localizing subcategory of  $K(\text{Inj } A)$ , generated by all compact objects  $X \in K(\text{Inj } A)$  such that  $LX$  is compact in  $D(\text{Mod } A)$ . Then we have that  $L_{\mathcal{K}} : \mathcal{K} \rightarrow D(\text{Mod } A)$  is an equivalence. Fix a left adjoint  $Q : D(\text{Mod } A) \rightarrow \mathcal{K}$ , the composition  $L_\lambda : D(\text{Mod } A) \xrightarrow{Q} \mathcal{K} \xrightarrow{inc} K(\text{Inj } A)$  is a left adjoint of  $L$ .

The fully faithful functor  $L_\lambda : D(\text{Mod } A) \rightarrow K(\text{Inj } A)$  identifies  $D(\text{Mod } A)$  with the localizing subcategory of  $K(\text{Inj } A)$  which is generated by all compact objects in the image of  $L_\lambda$ . That means the functor  $L_\lambda$  identifies  $D(A)$  with the localizing subcategory of  $K(\text{Inj } A)$  which is generated by the injective resolution  $I_A$  of  $A$ .

If the global dimension of  $A$  is finite then  $D(\text{Mod } A) \cong K(\text{Inj } A)$ . In this case,  $K(\text{Inj } A)$  is generically trivial if and only if  $D(\text{Mod } A)$  is generically trivial.

If the global dimension of  $A$  is infinite, then we can view  $D(\text{Mod } A)$  as a localizing subcategory of  $K(\text{Inj } A)$  by the adjoint functors. In this case, if  $X$  is an endofinite object in  $K(\text{Inj } A)$  then the object  $L(X)$  in  $D(\text{Mod } A)$  is endofinite by Lemma 2.2. Conversely, for an endofinite object  $Y \in D(\text{Mod } A)$ , we do not know whether  $L_\rho(Y)$  or  $L_\lambda(Y)$  is endofinite.

### 3. GENERICALLY TRIVIAL TRIANGULATED CATEGORIES

In this section, we introduce the generic object in a compactly generated triangulated category. We reformulate that a self-injective algebra  $A$  is of finite representation type by the stable module category  $\underline{\text{Mod}} A$  being generically trivial.

*Definition 3.1.* Let  $\mathcal{T}$  be a compactly generated triangulated category, an object  $E \in \mathcal{T}$  is called generic if it is an indecomposable endofinite object and not compact. The category  $\mathcal{T}$  is called generically trivial if it does not have any generic objects.

We give some examples to show the generic objects in derived categories. By the result in [12], the category  $D(\text{Mod } A)$  is generically trivial iff  $A$  is derived hereditary of Dynkin type.

*Example 1.* Let  $A$  be a finite dimensional  $k$ -algebra. The category  $D(\text{Mod } A)$  is a compactly generated triangulated category. If there exists a generic module  $M \in \text{Mod } A$ , then  $M$  viewed as a complex concentrated in one degree, which is a generic object in  $D(\text{Mod } A)$ .

For a self-injective algebra  $A$ , the stable category  $\underline{\text{Mod}} A$  is a compactly generated with  $\underline{\text{mod}} A$  the subcategory of compact objects. An  $A$ -module  $M$  is *endofinite* if it is finite length as  $\text{End}_A M$ -module [9]. The following result characterizes the endofinite object in the stable module category of a self-injective algebra.

*Lemma 3.2.* [6, Proposition 2.1] Let  $A$  be a finite dimensional self-injective algebra. The following conditions are equivalent.

- (1)  $M$  is an endofinite module in  $\text{Mod } A$ .
- (2)  $M$  is an endofinite object in  $\underline{\text{Mod}} A$ ,
- (3)  $\underline{\text{Hom}}_A(S, X)$  is finite length over  $\underline{\text{End}}_A X$  for every simple  $A$ -module  $S$ .

*Proposition 3.3.* Let  $A$  be a finite dimensional self-injective algebra. Then  $\underline{\text{Mod}} A$  is generically trivial if and only if  $A$  is of finite representation type.

*Proof.* If  $A$  is a self-injective algebra of finite type then every module is a direct sum of indecomposable finitely generated  $A$ -modules. Thus every  $A$ -modules is endofinite and each indecomposable endofinite module is finitely generated. Thus each endofinite object in  $\underline{\text{Mod}} A$  lies in  $\underline{\text{mod}} A$ . It implies that  $\underline{\text{Mod}} A$  is generically trivial.

Conversely, we assume that  $\underline{\text{Mod}} A$  is generically trivial. If  $A$  is representation infinite, then there exist a generic  $A$ -module  $G$ . By Lemma 3.2, we have that  $G$  is an generic object in  $\underline{\text{Mod}} A$ . This leads to a contradiction.  $\square$

#### 4. GENERIC OBJECTS IN $K(\text{Inj } A)$

In this section, we consider the generic object in  $K(\text{Inj } A)$  and give a criterion for  $K(\text{Inj } A)$  being generically trivial.

Recall that the singularity category  $D_{sg}^b(\text{mod } A)$  of an algebra  $A$  is the Verdier quotient of  $D^b(\text{mod } A)$  by  $K^b(\text{proj } A)$

$$D_{sg}^b(\text{mod } A) = D^b(\text{mod } A)/K^b(\text{proj } A).$$

By [14, Corollary 5.4], we have an equivalence up to direct factors  $\Gamma : D_{sg}^b(\text{mod } A) \rightarrow K_{ac}^c(\text{Inj } A)$ , i.e  $\Gamma$  is fully faithful and every object in  $K_{ac}^c(\text{Inj } A)^c$  is a direct factor of some objects in the image of  $\Gamma$ . The triangulated category  $\mathcal{D}$  is *Hom-finite* if the space  $\text{Hom}_{\mathcal{D}}(X, Y)$  is finite dimensional over  $k$  for each pair  $X, Y$  of objects in  $\mathcal{D}$ . The category  $D_{sg}^b(A)$  vanishes if and only if the global dimension of  $A$  is finite, denoted by  $gl.\dim A < \infty$ . However,  $D_{sg}^b(A)$  is not Hom-finite in general. In [8], there is a criterion for the hom-finiteness of  $D_{sg}^b(\text{mod } A)$  of a radical square zero Artin algebra  $A$ . We know that  $D_{sg}^b(A)$  is Hom-finite for a Gorenstein algebra  $A$ , i.e the injective dimension of  $A$  as  $A$ -module is finite.

Recall that an algebra  $A$  is *derived discrete* if and only if  $A$  is derived equivalent to an algebra of Dynkin type or a gentle algebra with one cycle in its quiver not satisfying the clock condition [17]. Each derived discrete algebra is a Gorenstein algebra [11]. It follows that  $K(\text{Inj } A)$  contains generic objects if  $A$  is derived discrete and not derived hereditary of Dynkin type by the following lemma.

*Lemma 4.1.* Assume that the global dimension of  $A$  is infinity. If the singularity category  $D_{sg}^b(A)$  of  $A$  is Hom-finite, then there exists generic object in  $K(\text{Inj } A)$ .

*Proof.* By the assumption, the category  $K_{ac}(\text{Inj } A)^c$  is Hom-finite. Given an object  $X \in K_{ac}(\text{Inj } A)$ , we have that

$$\text{Hom}_{K(\text{Inj } A)}(C, X) \cong \text{Hom}_{K_{ac}(\text{Inj } A)}(I_\lambda C, X), \quad \text{for } C \in K(\text{Inj } A)^c. \quad (\dagger)$$

The left adjoint functor  $I_\lambda$  preserves compact objects. It follows that  $I_\lambda(C) \in K_{ac}(\text{Inj } A)^c$ . We choose an indecomposable object  $Z \in K_{ac}(\text{Inj } A)^c$  which is not compact in  $K(\text{Inj } A)$ . In order to show that  $Z$  is a generic object in  $K(\text{Inj } A)$ , we only need to check that  $Z$  is an endofinite object in  $K(\text{Inj } A)$ . It follows from that the isomorphism  $(\dagger)$  and  $K_{ac}(\text{Inj } A)^c$  is Hom-finite.  $\square$

*Lemma 4.2.* If  $A$  is of infinite representation type, and  $M$  is a generic  $A$ -module, then the minimal injective resolution  $I_M$  is a generic object in  $K(\text{Inj } A)$ .

*Proof.* We need to show that  $\text{Hom}(C, I_M)$  is finite length over  $\text{End}_{K(\text{Inj } A)}(I_M)$  for all compact objects  $C$  in  $K(\text{Inj } A)$ , where  $I_M$  is a injective resolution of an  $A$ -module  $M$ . It suffices to consider the isomorphism

$$\text{Hom}_{K(\text{Inj } A)}(I_S, I_M) \cong \text{Hom}_{D(A)}(I_S, M) \cong \text{Hom}_{D(A)}(S, M) \cong \text{Hom}_A(S, M),$$

for any simple  $A$ -module  $S$ .

For a projective resolution of  $S$

$$\dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0 \dots,$$

we obtain the following exact sequence by applying the functor  $\text{Hom}_A(-, M)$  to the resolution of  $S$ ,

$$0 \rightarrow \text{Hom}_A(S, M) \rightarrow \text{Hom}_A(P_0, M) \rightarrow \text{Hom}_A(P_1, M) \rightarrow \dots$$

Since  $\text{Hom}_{D(\text{Mod } A)}(P_0, M)$  is finite length over  $\text{End}_{D(\text{Mod } A)} M \cong \text{End}_{K(\text{Inj } A)}(I_M)$ , thus  $\text{Hom}_A(S, M) \cong \text{Hom}_{K(\text{Inj } A)}(I_S, I_M)$  is finite length as  $\text{End}_{K(\text{Inj } A)}(I_M)$ -module.  $\square$

A finite dimensional  $k$ -algebra  $A$  is *derived endo-discrete* if for each sequence  $\{h_i\}_{i \in \mathbb{Z}}$  of non-negative integers there are only finitely many indecomposable object  $X$  in  $D^b(\text{mod } A)$  with

$$\text{length}_{\text{End}_{D^b(A)}(X)}(H^i(X)) = h_i$$

for all  $i \in \mathbb{Z}$ . For the field  $k$  with infinite cardinality,  $A$  is derived discrete if and only if  $A$  is derived endo-discrete [4, Theorem 1.1]. In this case, there exists a generic object in  $D^b(\text{Mod } A)$  if and only if  $A$  is not derived discrete.

The definition of generic object in [4] is not as same as the Definition 3.1. The main different is that the triangulated category  $D^b(\text{Mod } A)$  is not compactly generated. If there is a generic object  $X$  in  $D^b(\text{Mod } A)$ , then  $X$  is a generic object in  $D(\text{Mod } A)$  which is compactly generated. Indeed, the object  $X$  is an endofinite object in  $D(\text{Mod } A)$  by [10, Lemma 7.1]. On the other hand,  $X$  is indecomposable and not in  $D^b(\text{mod } A)$ . Thus  $X$  is an generic object in  $D(\text{Mod } A)$ . We have the following result for the generic object in  $K(\text{Inj } A)$ .

*Proposition 4.3.* If  $K(\text{Inj } A)$  is generically trivial, then the algebra  $A$  is derived discrete.

*Proof.* Assume that  $A$  is not derived discrete, then  $A$  is not derived endo-discrete. Thus there exists a generic object  $M$  in  $D^b(\text{Mod } A)$ . The object  $M$  is indecomposable and not in  $D^b(\text{mod } A)$ . Moreover,  $M$  is an endofinite object in  $D(\text{Mod } A)$ . With the same argument in Lemma 4.2, the minimal injective resolution  $I_M$  of  $M$  is a generic object in  $K(\text{Inj } A)$ .  $\square$

*Proposition 4.4.* Assume  $A = kQ/I$  is an algebra with its quiver  $Q$  connected.

- (1) If  $gl.\dim A < \infty$ , then  $K(\text{Inj } A)$  is generically trivial if and only if  $A$  is derived hereditary of Dynkin type.
- (2) Assume that  $A$  is Gorenstein with  $gl.\dim A = \infty$ , then  $K(\text{Inj } A)$  has generic objects.

*Proof.* (1) If  $gl.\dim A < \infty$ , then  $K(\text{Inj } A)$  is equivalent to  $D(\text{Mod } A)$  as triangulated categories. By the definition, generic objects are invariant under triangulated equivalence. By [12, Theorem 3.11],  $D(\text{Mod } A)$  is generically trivial if and only if  $A$  is derived hereditary of Dynkin type. Generical objects are invariant under an equivalence of categories.

(2) If  $A$  is a Gorenstein algebra, then  $D_{sg}^b(A)$  is Hom-finite. The result follows from Lemma 4.1.  $\square$

We summarize the above results in the following theorem.

*Theorem 4.5.* The followings are equivalent:

- (1) The categories  $K(\text{Inj } A)$  is generically trivial.
- (2) The categories  $D(\text{Mod } A)$  is generically trivial.
- (3) The algebra  $A$  is derived equivalent to an algebra of Dynkin type.

*Proof.* (2)  $\Leftrightarrow$  (3) follows from [12, Theorem 3.11]. We show that (1)  $\Leftrightarrow$  (3). It is trivial to show that (3)  $\Rightarrow$  (1). Now, if  $K(\text{Inj } A)$  is generically trivial, then the algebra  $A$  is derived discrete by Proposition 4.3. Assume that  $A$  is not derived equivalent to an algebra of Dynkin type, then it is derived equivalent to a gentle algebra with one cycle, which is a Gorenstein algebra. By Proposition 4.4, there exist generic objects in  $K(\text{Inj } A)$ . It is contradiction. Thus  $A$  is derived equivalent to an algebra of Dynkin type.  $\square$

## 5. LOCALLY FINITE TRIANGULATED CATEGORIES AND KRULL-GABRIEL DIMENSION

In this section, we assume that triangulated categories are small.

5.1. For a Hom-finite triangulated category  $\mathcal{D}$ , the category  $\mathcal{D}$  is *locally finite* if for each indecomposable object  $X$  of  $\mathcal{D}$ , there are only finitely many isoclasses of indecomposable objects  $Y$  such that  $\text{Hom}_{\mathcal{D}}(X, Y) \neq 0$  [18].

Let  $\text{Mod } \mathcal{D}$  be the category of all additive functors  $F : \mathcal{D}^{op} \rightarrow \text{Ab}$  into the category of abelian groups. A functor  $F \in \text{Mod } \mathcal{D}$  is called *finitely presented* if there exists an exact sequence

$$\text{Hom}_{\mathcal{D}}(-, X) \rightarrow \text{Hom}_{\mathcal{D}}(-, Y) \rightarrow F \rightarrow 0,$$

for  $X$  and  $Y$  in  $\mathcal{D}$ . The *abelianisation*  $Ab(\mathcal{D})$  of a triangulated category  $\mathcal{D}$  is the full subcategory of  $\text{Mod } \mathcal{D}$  consisting of all finitely presented objects. There is an characterization of locally finite triangulated category due to Auslander [2, Theorem 2.12] or refer to [15, Proposition 2.3].

*Lemma 5.1.* An triangulated category  $\mathcal{D}$  with split idempotents is locally finite if and only if for each object  $Y$  the following holds.

- (1) There are only finite many isomorphism classes of indecomposable objects  $X$  satisfying  $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ .
- (2) For each indecomposable object  $X$ ,  $\text{Hom}_{\mathcal{D}}(X, Y)$  is finite length as  $\text{End}_{\mathcal{D}} X$ -module.

There is a list of locally finite triangulated categories in [15], and we show an example in this list.

*Example 2.* Let  $A$  be a finite dimensional self-injective  $k$ -algebra. Then  $\underline{\text{mod}} A$  is locally finite if and only if  $A$  is of finite representation type. At the same time,  $\underline{\text{mod}} A$  is locally finite if and only if  $\underline{\text{Mod}} A$  is generically trivial.

5.2. Let  $\mathcal{A}$  be an abelian category, a subcategory  $\mathcal{A}'$  of  $\mathcal{A}$  is called *Serre subcategory* if it is closed under subobjects, quotients and extensions. Then one can define the quotient subcategory  $\mathcal{A}/\mathcal{A}'$  as follows. The objects in  $\mathcal{A}/\mathcal{A}'$  coincides with the the objects in  $\mathcal{A}$ . The morphisms are defined by

$$\text{Hom}_{\mathcal{A}/\mathcal{A}'}(X, Y) = \varinjlim_{\substack{X' \subseteq X, Y' \subseteq Y \\ X/X', Y'/Y \in \mathcal{A}'}} \text{Hom}_{\mathcal{A}}(X', Y/Y').$$

There is a natural quotient functor  $q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}'$ . Set  $\mathcal{A}_{-1} = 0$ , and for each  $n \in \mathbb{Z}_{>0}$ ,  $\mathcal{A}_n$  is the subcategory of objects  $X$  in  $\mathcal{A}$  which are finite length in  $\mathcal{A}/\mathcal{A}_{n-1}$  under the quotient functor  $q$ . The *Krull-Gabriel dimension*  $\text{KG dim } \mathcal{A}$  of  $\mathcal{A}$  is the smallest integer  $n$  such that  $\mathcal{A} = \mathcal{A}_n$ .

Krull-Gabriel dimension measures how far  $\text{Ab}(\mathcal{D})$  is from being a length category. One has that  $\text{KG dim } \text{Ab}(\mathcal{D}) = 0$  iff  $\mathcal{D}$  is locally finite. For example,  $\mathcal{A} = \text{Ab}(\text{mod } A)$  for some finite dimensional  $k$ -algebra, then  $\text{KG dim } \mathcal{A} = 0$  iff  $A$  is of finite representation type [2]. In the cases of triangulated categories, we have the similar result. For  $\mathcal{T} = D(\text{Mod } A)$  or  $\mathcal{T} = K(\text{Inj } A)$ , the category  $\mathcal{T}^c$  is locally finite if and only if  $\text{KG dim } \text{Ab}(\mathcal{T}^c) = 0$  [7, Main theorem]. In general, a category  $\mathcal{D}$  is locally finite if and only if  $\text{Ab}(\mathcal{D})$  is a length category.

*Lemma 5.2.* Let  $\mathcal{D}$  be a small triangulated category and idempotent split. The category  $\mathcal{D}$  is locally finite if and only if  $\text{KG dim}(\text{Ab}(\mathcal{D})) = 0$ .

*Proof.* If an essential small triangulated category  $\mathcal{D}$  is locally finite, then each representable functor  $\text{Hom}_{\mathcal{D}}(-, X)$  is of finite length. For each  $F \in \text{Ab}(\mathcal{D})$ , there are only finitely many  $X \in \mathcal{D}$  such that  $F(X) \neq 0$ . The functor is finite length by [3, Corollary 3.11]. That means  $\text{KG dim } \text{Ab}(\mathcal{D}) = 0$ .

Conversely, if  $\text{KG dim } \text{Ab}(\mathcal{D}) = 0$ , then each functor  $\text{Hom}(-, X)$  is finite length. By the equivalence between  $\text{Ab}(\mathcal{D}^{op})$  and  $\text{Ab}(\mathcal{D})^{op}$  induced by  $\text{Hom}_{\mathcal{D}}(-, X)$  to  $\text{Hom}_{\mathcal{D}}(X, -)$ . Thus  $\text{Hom}(-, X)$  and  $\text{Hom}(X, -)$  both are finite length functor for any  $X \in \mathcal{D}$ . It follows that  $\mathcal{D}$  is a locally finite category.  $\square$

The abelianisation  $\text{Ab } \mathcal{T}^c$  is called *noetherian* if it satisfies the ascending chain condition on subobjects. This is equivalent to the category  $\text{Mod } \mathcal{T}^c$  has a set of generators consisting of noetherian objects.

We say a compactly generated triangulated category  $\mathcal{T}$  is *associated with an algebra*  $A$  if it is of the form either  $K(\text{Inj } A)$  or  $D(\text{Mod } A)$  or  $\underline{\text{Mod}} A$  with  $A$  a self-injective algebra. One has the following result about generically trivial triangulated categories.

*Theorem 5.3.* Let  $\mathcal{T}$  be a compactly generated triangulated category associated with an algebra  $A$ , set  $\mathcal{A} = \text{Ab}(\mathcal{T}^c)$  then the following are equivalent.

- (1)  $\mathcal{T}$  is generically trivial.
- (2)  $\mathcal{T}^c$  is locally finite.
- (3)  $\text{KG dim } \mathcal{A} = 0$ .
- (4)  $\mathcal{A}$  is noetherian.

*Proof.* If  $\mathcal{T} = \underline{\text{Mod}} A$  for a self-injective algebra  $A$ , then  $\mathcal{T}$  is generically trivial iff  $A$  is of finite type Proposition 3.3. Thus all the statements are equivalent.

Now we consider  $\mathcal{T}$  is of the form  $K(\text{Inj } A)$  or  $D(\text{Mod } A)$  for an algebra  $A$ . The equivalence (2)  $\Leftrightarrow$  (3) follows from Lemma 5.2. The equivalence (1)  $\Leftrightarrow$  (2) follows from Theorem 4.5. The equivalence (1)  $\Leftrightarrow$  (4) follows from Theorem 4.5 and [5, Theorem 12.20].  $\square$

For a compactly generated triangulated category  $\mathcal{T}$ , we say that  $\mathcal{T}$  is of *finite type* if  $\text{Mod } \mathcal{T}^c$  has a set of generators of finite length [5, Definition 11.19]. Combine [5, Proposition 11.23] and Theorem 5.3, we have the following result.

*Corollary 5.4.* Let  $\mathcal{T}$  be a compactly generated triangulated category associated with an algebra  $A$ , set  $\mathcal{A} = \text{Ab}(\mathcal{T}^c)$  then the following are equivalent.

- (1)  $\mathcal{T}$  is of finite type.
- (2)  $\text{KG dim } \text{Ab}(\mathcal{T}) = 0$ .
- (3)  $\mathcal{T}$  is generically trivial.
- (4)  $\mathcal{T}^c$  is locally finite.

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